

# Design of controllers for a Non-linear system using Pole-placement and Linear Quadratic Regulator Technique to solve Servo and Regulatory problem

Neetha R

**Abstract**— This project mainly discusses about the design of controller for a nonlinear system like rotary inverted pendulum. This system is under actuated and well-suited for verification and practice of ideas emerging in control theory. Nonlinear systems exhibit instability, making the design of controllers for balancing in the stable position, a challenging problem. The dynamic model of the rotary inverted pendulum is identified first and the state space representation of the system is obtained. The controller is designed by using the pole placement technique and Linear Quadratic Regulator technique in MATLAB software package. The regulatory problem and the servo problem of rotary inverted pendulum system is solved with these controllers designed. The different controller design is implemented in simulation and their performances are compared. Simulation results onto a nonlinear system are given to illustrate the effectiveness of the developed strategies.

**Index Terms**— Non-Linear Systems, Rotary Inverted Pendulum, Pole Placement Technique, Servo Problem, Regulatory problem.

## I. INTRODUCTION

ROTARY INVERTED PENDULUM represents a significant class of nonlinear under actuated mechanical systems, well-suited for verification and practice of ideas emerging in control theory. It is a very good model for the altitude control of a space booster rocket and a satellite, an automatic aircraft landing system, aircraft stabilization in the turbulent air-flow, stabilization of a cabin in a ship etc.

Stabilization of a pendulum rod in the unstable upright position is considered a benchmark control problem which has been solved by attaching the pendulum to a base that moves in a rotary manner in a horizontal plane.

Rotary inverted pendulum is excited by a DC motor, and is equipped with sensors to measure the angular displacement of the pendulum and the angular velocity and position of the DC motor. It consists of an arm with a pivot at one end and a metal shaft on the other end. The pivot-end is mounted on top of the rotary servo base unit. The pendulum link is attached to a metal shaft and is instrumented with an encoder to measure its angle. The result is a horizontally rotating arm with a pendulum at the end. The paper is organized as follows:

- The modeling of the system and state space representation of the system is obtained.
- Control strategy which involves using pole-placement technique to solve the regulatory problem and LQR technique to solve the servo problem.

## II. MODELING OF THE ROTARY INVERTED PENDULUM

The rotary inverted pendulum consists of three sections namely, the arm section, the pendulum bob section and the motor section. The standard right-handed Cartesian co-ordinate system is used. Fig1 shows the simple rotary inverted pendulum and fig2 shows the co-ordinate system for the derivation of the dynamic model of the system.



Fig1. Rotary inverted pendulum

The angular position of the arm,  $\alpha$ , is assigned to be increasing when the arm is rotating about the z-axis in the right-handed sense. The reference of  $\alpha$  is taken to be the x-axis. The angular position of the pendulum,  $\beta$  and  $\theta$ , are assigned to be increasing when the pendulum is rotating about an axis through the arm section from the origin to the pivot point of the pendulum, in the right-handed sense. The reference of  $\beta$  is taken from the upward vertical and the reference of  $\theta$  is taken from the downward vertical.

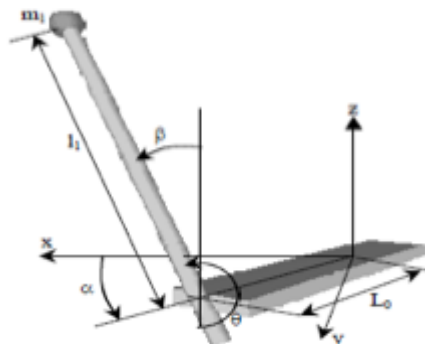


Fig2. co-ordinate system for derivation of dynamic model

### A. Equations involved in deriving the model:

The torque equation of the motor can be written as follows:

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$$\ddot{\theta} = \frac{1}{J_0} \ddot{\theta} - \frac{1}{J_0} \ddot{\theta}$$

The kinetic energy of the arm ,T<sub>0</sub>, and the potential energy ,V<sub>0</sub>, can be written as:

$$T_0 = \frac{1}{2} J_0 \dot{\theta}^2 \dots\dots\dots(1)$$

$$V_0 = 0$$

Kinetic energy of the pendulum bob , T<sub>1</sub> , and the potential energy , V<sub>1</sub>, can be written as:

$$T_1 = \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_1 (\dot{\theta}_1^2 + \dot{\theta}_1^2 + \dot{\theta}_1^2)$$

$$V_1 = \frac{1}{2} J_1 \dot{\theta}_0^2 + \frac{1}{2} (J_1 + J_1 \dot{\theta}_1^2) \dot{\theta}^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 \dot{\theta}^2 \dots\dots\dots(2)$$

$$T_1 = \frac{1}{2} J_1 \dot{\theta}_0^2 + \frac{1}{2} (J_1 + J_1 \dot{\theta}_1^2) \dot{\theta}^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 \dot{\theta}^2$$

$$V_1 = \frac{1}{2} J_1 \dot{\theta}_0^2 + \frac{1}{2} (J_1 + J_1 \dot{\theta}_1^2) \dot{\theta}^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 \dot{\theta}^2$$

The energy equation used in Lagrange's equation , L , can be formulated as:

$$L = T_0 - V_0$$

$$L = T_1 - V_1$$

$$L = T_0 + T_1 - V_0 - V_1$$

Applying (1) and (2) in the above equation:

$$L = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} J_1 \dot{\theta}_0^2 + \frac{1}{2} (J_1 + J_1 \dot{\theta}_1^2) \dot{\theta}^2$$

$$L = \frac{1}{2} (J_0 + J_1 \dot{\theta}_0^2) \dot{\theta}^2 + \frac{1}{2} (J_1 + J_1 \dot{\theta}_1^2) \dot{\theta}^2$$

$$L = \frac{1}{2} (J_0 + J_1 \dot{\theta}_0^2) \dot{\theta}^2 + \frac{1}{2} (J_1 + J_1 \dot{\theta}_1^2) \dot{\theta}^2$$

The equation for the axis of rotation for α , can be written as:

$$-\ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_1 \ddot{\theta}_1^2) \ddot{\theta} - \frac{1}{2} \ddot{\theta}_1 \ddot{\theta}_1^2 \ddot{\theta}^2 \ddot{\theta}_1 \ddot{\theta}_1$$

$$= -\ddot{\theta}_1 \ddot{\theta}_1$$

The equation for the axis of rotation for β , can be written as:

$$-\ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 \ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_1 \ddot{\theta}_1^2) \ddot{\theta} - \frac{1}{2} \ddot{\theta}_1 \ddot{\theta}_1^2 \ddot{\theta}^2 \ddot{\theta}_1 \ddot{\theta}_1$$

$$+ \ddot{\theta}_1 \ddot{\theta}_1$$

$$= 0$$

The dynamic equations for the upward position :

$$\ddot{\alpha} = \frac{1}{af - c^2} \begin{bmatrix} -df & ch & -cC_1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \frac{1}{af - c^2} \begin{bmatrix} ef \\ 0 \\ ce \end{bmatrix} u$$

$$\begin{bmatrix} J_0 + m_1 L_0^2 + m_1 l_1^2 \sin^2 \beta & -m_1 L_0 l_1 \cos \beta \\ -m_1 L_0 l_1 \cos \beta & J_1 + m_1 l_1^2 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} C_0 + \frac{1}{2} m_1 l_1^2 \dot{\beta} \sin 2\beta & -m_1 L_0 l_1 \dot{\beta} \sin \beta + \frac{1}{2} m_1 l_1^2 \dot{\alpha} \sin 2\beta \\ -\frac{1}{2} m_1 l_1^2 \dot{\alpha} \sin 2\beta & C_1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} 0 \\ m_1 g l_1 \sin \beta \end{bmatrix} = \begin{bmatrix} \tau_m \\ 0 \end{bmatrix}$$

The dynamic equations for the downward position:

$$\begin{bmatrix} J_0 + m_1 L_0^2 + m_1 l_1^2 \sin^2 \beta & -m_1 L_0 l_1 \cos \beta \\ -m_1 L_0 l_1 \cos \beta & J_1 + m_1 l_1^2 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} C_0 + \frac{K_t K_b}{R_a} \frac{1}{2} m_1 l_1^2 \dot{\beta} \sin 2\beta & -m_1 L_0 l_1 \dot{\beta} \sin \beta + \frac{1}{2} m_1 l_1^2 \dot{\alpha} \sin 2\beta \\ -\frac{1}{2} m_1 l_1^2 \dot{\alpha} \sin 2\beta & C_1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} 0 \\ m_1 g l_1 \sin \beta \end{bmatrix} = \begin{bmatrix} \frac{K_t K_u}{R_a} \\ 0 \end{bmatrix} u$$

## B. Representation

### Symbol Unit Description

- α rad arm angle about z-axis, taken from positive x-axis.
- β rad pendulum angle, taken from the upward vertical axis.
- θ rad pendulum angle, taken from the downward vertical axis.
- J<sub>0</sub> Kg- m<sup>2</sup> moment of inertia of arm section
- J<sub>1</sub> Kg- m<sup>2</sup> moment of inertia of pendulum section
- C<sub>0</sub> Kg- m<sup>2</sup>/s viscous friction co-efficient of arm section
- C Kg- m<sup>2</sup>/s viscous friction co-efficient of pendulum section
- m<sub>1</sub> Kg effective mass of the pendulum section
- l<sub>1</sub> m effective length of pendulum section
- K<sub>t</sub> N- m/A motor torque constant
- K<sub>b</sub> V- s/ Rad motor back EMF constant
- K<sub>u</sub> V/count motor back EMF constant
- R<sub>a</sub> Ω motor armature coil resistance
- g m/s<sup>2</sup> acceleration due to gravity
- u counts motor driving command
- L<sub>0</sub> m physical distance between the pivot of pendulum section and axis of rotation of arm section

### C. State Space Representation

The linear model can be represented as:

where

$$a = \ddot{\theta}_0 + \ddot{\theta}_1 \ddot{\theta}_0^2$$

$$b = \ddot{\theta}_1 \ddot{\theta}_1^2$$

Using the pole placement technique, the gain value of the controller is found as:  $K = 3.0108 \ 159.8903 \ 10.9588$

The plots of arm angle versus time and pendulum angle taken from upward vertical axis versus time, and pendulum angle taken from downward vertical axis versus time are as follows:

$$c = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}{\begin{bmatrix} 1 \end{bmatrix}}$$

$$e = \frac{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}{\begin{bmatrix} 1 \end{bmatrix}}$$

$$f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2$$

and

$$J_0 = 0.08591 \text{ (kg m}^2\text{)}$$

$$J_1 = 0.000217 \text{ (kg m}^2\text{)}$$

$$J_0 = 0.006408 \text{ (kg m}^2\text{/s)}$$

$$J_1 = 0.000158 \text{ (kg m}^2\text{/s)}$$

$$K_t = 0.1144 \text{ (Nm/A)}$$

$$K_b = 0.1146 \text{ (Vs/Rad)}$$

$$R_a = 2.3 \text{ (}\Omega\text{)}$$

$$m_1 = 0.0319 \text{ (kg)}$$

$$l_1 = 0.1572 \text{ (m)}$$

$$L_0 = 0.1370 \text{ (m)}$$

$$K_u = 0.0523 \text{ (V/Count)}$$

$$g = 9.81 \text{ (m/s}^2\text{)}$$

Substituting the known constant values and applying them in the above equations, we get

$$A = \begin{bmatrix} -1.3705 & 3.8552 & -0.01242 \\ 0 & 0 & 1 \\ -0.9367 & 51.5747 & -0.1661 \end{bmatrix} \quad B = \begin{bmatrix} 0.5721 \\ 0 \\ 0.3910 \end{bmatrix}$$

### III. REGULATORY PROBLEM

In this case, the desired value is to remain fixed and the purpose of the control system is to maintain the controlled variable at desired value in spite of changes in load. The general block diagram of the regulatory problem of the rotary inverted pendulum is shown in the below figure.

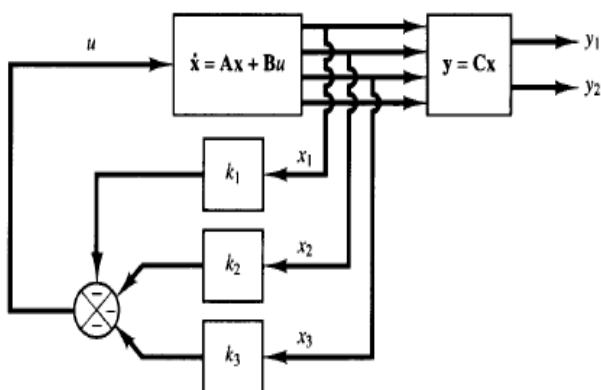


Fig3. Regulatory problem of rotary inverted pendulum

#### A. Pole-Placement Technique:

Actual poles = eig(A) = -7.3083 -1.2981 7.0698

Desired poles are calculated by keeping the settling time as 5 seconds and peak overshoot as 1.

Desired poles = -0.8 + 2.25i -0.8 - 2.25i -2

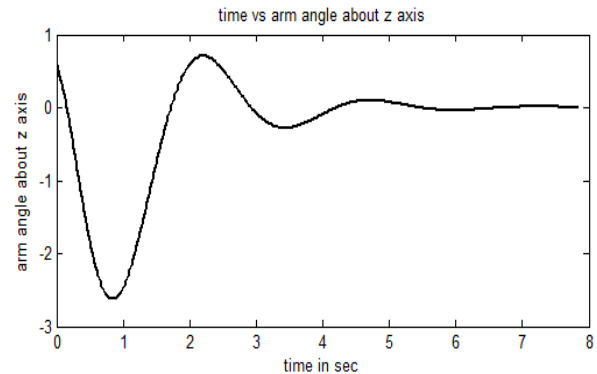


Fig4. time vs arm angle about z-axis

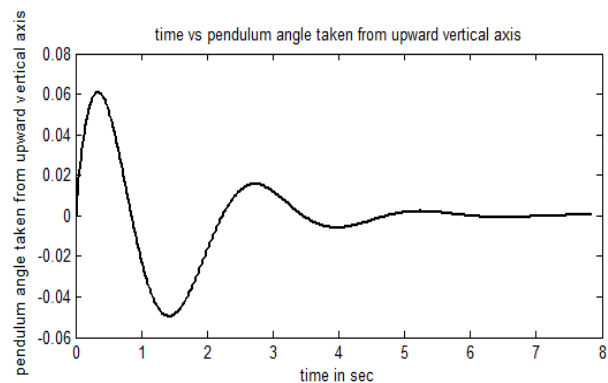


Fig5. time vs pendulum angle taken from upward vertical axis

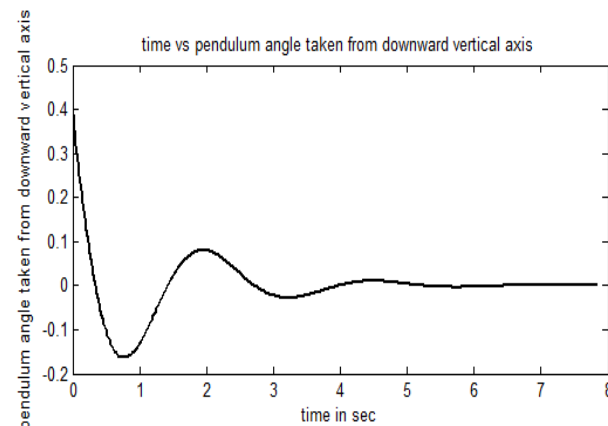


Fig6. time versus pendulum angle taken from downward vertical axis

### IV. SERVO PROBLEM

In this case, we assume that there is no change in load and we are interested in changing the variable according to some function of time.

The servo problem can be solved using the LQR technique

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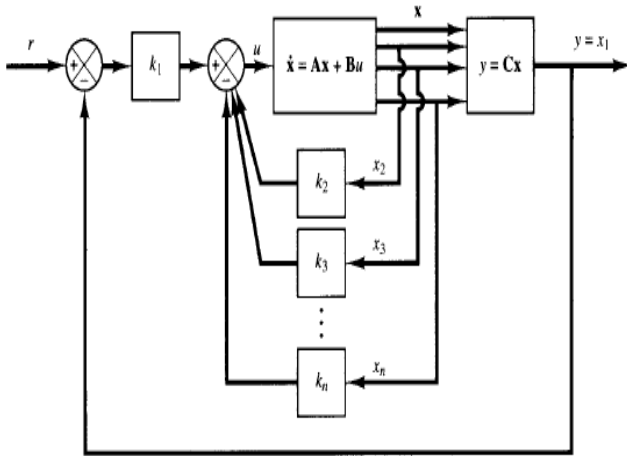


Fig7.type1 servo system

## D.Linear Quadratic Regulator Technique



$$\dot{x} = Ax + Bu$$

Where x= state vector(n-vector)

u=control vector(r-vector)

A=n\*n constant matrix

B=n\*r constant matrix

A quadratic performance index,where the limits of the integration are 0 and infinity, such as

$$J = \int_0^{\infty} L(x,u)dt$$

Where L(x,u) is a quadratic function or Hermitian function of x and u , will yield linear control laws; that is

$$u(t) = -Kx(t)$$

The design of optimal regulator systems based on such quadratic performance indexes boils down to the determination of the elements of the matrix K.

$$J = \int_0^{\infty} (x^* Q x + u^* R u) dt$$

Where Q is the positive –definite Hermitian, or , real symmetric matrix ,R is a positive-definite hermitian or real symmetric matrix ,and u is unconstrained.

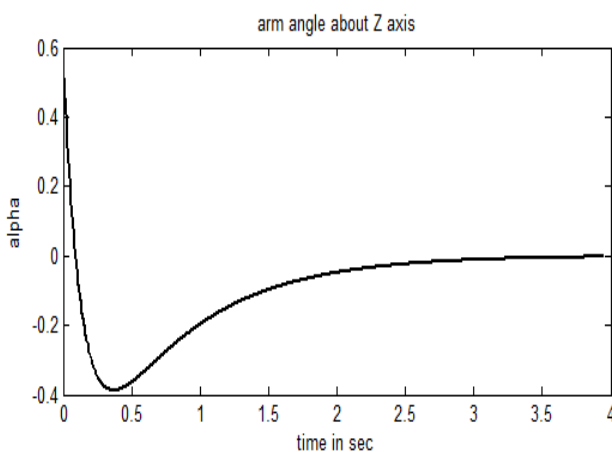


Fig8. time vs arm angle about z-axis

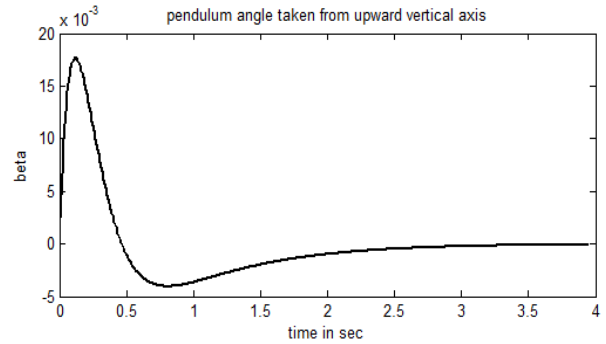


Fig9. time vs pendulum angle taken from upward vertical axis

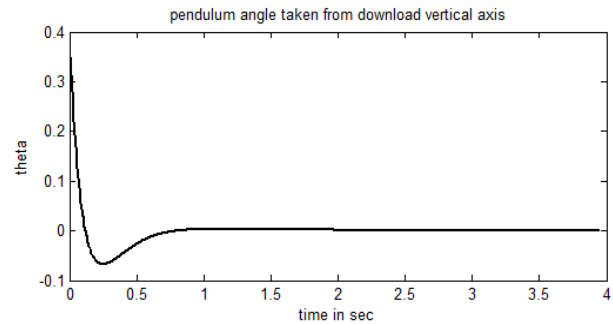


Fig10. time vs pendulum angle taken from downward vertical axis

## V. DISCUSSION AND RESULTS

Earlier, the nonlinear systems and their dynamic equations were understood by obtaining their transfer functions and designing a suitable controller using the transfer function. The state space representation is a new technique and it also helps in understanding the interaction of one state variable with another. It also explains how each input has an effect on the output. Thus the rotary inverted pendulum system can be linearized by observing the factors that influence the linearization process. The stabilization of the nonlinear systems was done through the pole placement technique. The actual poles of the nonlinear system were replaced with the desired poles by giving a desired settling time. Thus it was found that the system can be stabilized in spite of any change in loads. The rotary inverted pendulum system is stabilized about one equilibrium position i.e, the arm angle. Thus the rotary inverted pendulum system was able to track to a given set point. This servo problem was solved using the linear quadratic regulator technique. Though a lot of work has been done on these systems, solving the servo and regulatory problem is always a challenging work when a desired settling time is required.

## VI. CONCLUSION

In this paper, the state space representation of the nonlinear systems were obtained from the dynamic equations. Since a suitable controller to stabilize the system about one equilibrium point was designed using the pole placement technique proper gain values were obtained and the nonlinear systems were proved to be regulatory. Thus the rotary inverted pendulum system was able to track to a given set point.

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**Neetha R** did her undergraduation in Electronics and Instrumentation Engineering at the St.Joseph's College of Engineering. She received the engineering degree in 2015 and aspires to contribute to the field of Instrumentation engineering.